A version of the proof for Peres-Schlag's theorem on lacunary sequences.

Moshchevitin N.G. [1]

We present a proof of a multidimensional version of Peres-Schlag's theorem on Diophantine approximations with lacunary sequences.

1. Introduction.

A sequence $\{t_j\}$, $j = 1, 2, 3, ..$ of positive real numbers is defined to be lacunary if for some $M > 0$ one has

$$\frac{t_{j+1}}{t_j} \geq 1 + \frac{1}{M}, \ \ \forall j \in \mathbb{N}. \qquad (1)$$

Erdös [1] conjectured that for any lacunary sequence there exists real α such that the set of fractional parts $\{\alpha t_j\}$, $j \in \mathbb{N}$ is not dense in $[0, 1]$. This conjecture was proved by Pollington [2] and de Mathan [3]. Some quantitative improvements were due to Katznelson [4], Akhunzhanov and Moshchevitin [5] and Dubickas [6]. The best known quantitative estimate is due to Peres and Schlag [7]. The last authors proved that with some positive constant $\gamma > 0$ for any sequence $\{t_j\}$ under condition (1) there exists a real number α such that

$$||\alpha t_j|| \geq \frac{\gamma}{M \log M}, \ \ \forall j \in \mathbb{N}.$$

In their proof Peres and Schlag use a special variant of the Lovasz local lemma.

In the present paper we (following the arguments from [7]) prove a multidimensional version of the above result by Peres and Schlag. Our proofs avoid the terminology from probability theory.

2. Notation and parameters. $\mu(\cdot)$ denotes the Lebesque measure. For a set $A \subset \mathbb{R}^d$ we denote $A^c = [0, 1]^d \setminus A$. Let $M \geq 8$ and $t_1 \geq 2$. We shall

[1] Research is supported by grants RFFI 06-01-00518, MD-3003.2006.1, NSh-1312.2006.1 and INTAS 03-51-5070

use the parameters

$$\delta = \frac{1}{2^{11}(M\log M)^{1/d}}, \quad \delta_1 = 2^{4d+1}\delta^d, \quad h = \lceil 2^{3d} M \log M \rceil$$

and

$$l_j = \left\lfloor \log_2 \frac{t_j}{2\delta} \right\rfloor. \tag{2}$$

Note that

$$1 - (16\delta)^d h \geq 1/2, \tag{3}$$

$$1 - 2\delta_1 h \geq 1/2, \tag{4}$$

and from (1) one has

$$\frac{t_{i+h}}{t_i} \geq \left(1 + \frac{1}{M}\right)^h \geq M^{2^{3d}\log 2} \geq \frac{1}{\delta}. \tag{5}$$

For the proof of our result we need the sets

$$E(j,a) = \left\{ x \in [0,1] : \ \left| x - \frac{a}{t_j} \right| \leq \frac{\delta}{t_j} \right\}.$$

For integer point $\mathbf{a} = (a_1, ..., a_d) \in \mathbb{R}^d$ denote

$$E(j,\mathbf{a}) = E(j,a_1) \times ... \times E(j,a_d)$$

Each $E_{j,a}$ is covered by a opened dyadic segments of the form $\left(\frac{b}{2^{l_j}}, \frac{b+\varepsilon}{2^{l_j}}\right), \ \ \varepsilon \in \{1,2\}$. Then the set

$$\bigcup_{0 \leq a_1,...,a_d \leq \lceil t_j \rceil} E(j,\mathbf{a}) \bigcap [0,1]^d$$

is covered by some union of dyadic boxes of the form

$$\left(\frac{b_1}{2^{l_j}}, \frac{b_1+\varepsilon_1}{2^{l_j}}\right) \times ... \times \left(\frac{b_d}{2^{l_j}}, \frac{b_d+\varepsilon_d}{2^{l_j}}\right) \ \ \varepsilon_k \in \{1,2\}.$$

We denote this union as A_j. So

$$\bigcup_{0 \leq a_1,...,a_d \leq \lceil t_j \rceil} E(j,\mathbf{a}) \cap [0,1]^d \subseteq A_j,$$

and the complement $A_j^c = [0,1]^d \setminus A_j$ can be represented as a union $A_j^c = \cup_{1\le\nu\le T_j} I_\nu$ of closed dyadic boxes of the form

$$\left[\frac{b_1}{2^{l_j}}, \frac{b_1+1}{2^{l_j}}\right] \times ... \times \left[\frac{b_d}{2^{l_j}}, \frac{b_d+1}{2^{l_j}}\right]. \tag{6}$$

Moreover the set $\cap_{j\le i} A_j^c$ also can be represented as

$$\bigcap_{j\le i} A_j^c = \bigcup_{1\le\nu\le T_i} I_\nu$$

with dyadic intervals I_ν of the form (6). Note that

$$\mu(A_j) \le \left(\frac{4\delta}{t_j}\right)^d (\lceil t_j\rceil + 1)^d \le (16\delta)^d. \tag{7}$$

3. The results.

Theorem 1. *Let $d \in \mathbb{N}$ and $t_1 \ge 2,\ M \ge 8$. Then for any sequence $\{t_j\}$ under condition (1) there exists a set of real numbers $(\alpha_1, ..., \alpha_d)$ such that*

$$\max_{1\le k\le d} ||\alpha_k t_j|| \ge \frac{1}{2^{11}(M\log M)^{1/d}},\ \ \forall j \in \mathbb{N}.$$

We give the proof of the Theorem 1 in sections 4,5.

For a vector $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$ we define

$$||\xi|| = \max_{1\le j\le d} \min_{a_j\in\mathbb{Z}} |\xi_j - a_j|.$$

Theorem 2. *Let $d \in \mathbb{N}$ and $t_1 \ge 2,\ M \ge 8$. Let the sequence $\{t_j\}$ satisfies (1). Let $\{S_j\} \subset \mathrm{O}_d$ be a sequence of orthogonal matrices and $G_j = t_j S_j$. Then there exists a real vector $\alpha = (\alpha_1, ..., \alpha_d)$ such that*

$$||G_j\alpha|| \ge \frac{1}{2^{13}d(M\log M)^{1/d}},\ \ \forall j \in \mathbb{N}.$$

Of course the constants in Theorems 1,2 may be improved.

Corollary. *There exists an effective positive constant Δ with the following property. For any complex number $\theta = a+bi$, $|\theta| > 1$ there exists complex α such that for the the distance to the nearest Gaussian integer (in sup-norm) one has*

$$||\theta^j\xi|| \ge \Delta \min\left\{\frac{(|\theta|-1)}{\log(2+1/(|\theta|-1))}; 1\right\},\ \ \forall j \in \mathbb{N}.$$

The corollary follows from the Theorem 2 for $d = 2$ with lacunary sequence $|\theta^j|$ and the matrix sequence S^j, $S = \frac{1}{|\theta|}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. The case $|\theta| > 1$ easily can be reduced to the case $|\theta| > 8$.

We must note that recently Dubickas [8] proved the following result. Let $t_0, t_1, t_2, ...$ be a sequence of non-zero complex numbers satisfying $|t_{j+1}| \geq a|t_j|$ with real $a > 1$. Let νi be a complex number. Then there exists a complex number α such that the numbers $\alpha t_j, j = 0, 1, 2, ...$ all lie outside the union of open squares centered at $\nu + \mathbb{Z}[i]$ whose sides, parallel to real and imaginary axis are equal to $(a-1)/20$ for $a \in (1, 11-4\sqrt{5}]$ and $(a-2)/(a-1)$ for $a > 11 - 4\sqrt{5}$.

4. The principal lemma

Lemma 1. *Let* $\mu\left(\bigcap_{j\leq i} A_j^c\right) \neq 0$ *Then*

$$\mu\left(A_{i+h}\bigcap(\bigcap_{j\leq i} A_j^c)\right) \leq \delta_1 \mu\left(\bigcap_{j\leq i} A_j^c\right).$$

Proof. The proof is identical to the proof of formula (3.9) from [7]. We have

$$\mu\left(A_{i+h}\bigcap(\bigcap_{j\leq i} A_j^c)\right) = \sum_{\nu=1}^{T_i} \mu(A_{i+h}\bigcap I_\nu).$$

As $\mu\left(\cap_{j\leq i} A_j^c)\right) \neq 0$ the sum here is not empty and $T_i \geq 1$. A_{i+h} can be covered by a union of closed dyadic boxes of the form

$$\left[\frac{b_1}{2^{l_{j+h}}}, \frac{b_1+1}{2^{l_{j+h}}}\right] \times ... \times \left[\frac{b_d}{2^{l_{j+h}}}, \frac{b_d+1}{2^{l_{j+h}}}\right].$$

Let J be a dyadic box from this covering of A_{i+h} (its measure is equal to $2^{-dl_{i+h}}$). If $\mu(J \cap I_\nu) \neq 0$ then $J \subseteq I_\nu$. Let

$$I_\nu = \left[\frac{b_1}{2^{l_i}}, \frac{b_1+1}{2^{l_i}}\right] \times ... \times \left[\frac{b_d}{2^{l_i}}, \frac{b_d+1}{2^{l_i}}\right]$$

and the box J appears from the covering of the box $E(i+h, \mathbf{a})$. Then

$$\frac{\mathbf{a}}{t_{i+h}} \in \left(\frac{b_1}{2^{l_i}} - \frac{\delta}{t_{i+h}}, \frac{b_1+1}{2^{l_i}} + \frac{\delta}{t_{i+h}}\right) \times ... \times \left(\frac{b_d}{2^{l_i}} - \frac{\delta}{t_{i+h}}, \frac{b_d+1}{2^{l_i}} + \frac{\delta}{t_{i+h}}\right).$$

Let W_ν be the number of integer points $\mathbf{a}$ satisfying the condition above. Then

$$W_\nu \le \left(\left\lfloor \left(\frac{1}{2^{l_i}} + \frac{2\delta}{t_{i+h}} \right) t_{i+h} \right\rfloor + 1 \right)^d \le (2^{-l_i} t_{i+h} + 2)^d. \tag{8}$$

Now we deduce

$$\mu \left(A_{i+h} \bigcap \left(\bigcap_{j \le i} A_j^c \right) \right) \le 2^d \sum_{\nu=1}^{T_i} \frac{2^d W_\nu}{2^{dl_{i+h}}} \le \sum_{\nu=1}^{T_i} \frac{(2^{-l_i} t_{i+h} + 2)^d}{2^{dl_{i+h}}} \le \tag{9}$$

$$\le 2^{2d} \sum_{\nu=1}^{T_i} \left(\max \left(\frac{2^{-l_i} t_{i+h}}{2^{l_{i+h}}}, \frac{1}{2^{l_{i+h}-1}} \right) \right)^d \le 2^{2d} \left(\sum_{\nu=1}^{T_i} \mu(I_\nu) \left(\frac{t_{i+h}}{2^{l_{i+h}}} \right)^d + \sum_{\nu=1}^{T_i} \left(\frac{1}{2^{l_{i+h}-1}} \right)^d \right).$$

But

$$\sum_{\nu=1}^{T_i} \mu(I_\nu) \left(\frac{t_{i+h}}{2^{l_{i+h}}} \right)^d = \mu \left(\bigcap_{j \le i} A_j^c \right) \left(\frac{t_{i+h}}{2^{l_{i+h}}} \right)^d.$$

Applying (2) we have $l_{i+h} \ge \log_2 \frac{t_{i+h}}{2\delta} - 1$ and

$$\sum_{\nu=1}^{T_i} \mu(I_\nu) \left(\frac{t_{i+h}}{2^{l_{i+h}}} \right)^d \le (4\delta)^d \mu \left(\bigcap_{j \le i} A_j^c \right). \tag{10}$$

From another hand

$$\sum_{\nu=1}^{T_i} \left(\frac{1}{2^{l_{i+h}-1}} \right)^d = \frac{2^d T_i}{2^{dl_{i+h}}} = 2^d \mu \left(\bigcap_{j \le i} A_j^c \right) \left(\frac{2^{l_i}}{2^{l_{i+h}}} \right)^d$$

as $\mu \left(\bigcap_{j \le i} A_j^c \right) = T_i 2^{-dl_i}$. Now from (2) we have $\frac{2^{l_i}}{2^{l_{i+h}}} \le 2 \frac{t_i}{t_{i+h}}$. Applying (5) we deduce

$$\sum_{\nu\left(\bigcap_{j \le i} A_j^c\right)=1}^{T_i} \left(\frac{1}{2^{l_{i+h}-1}} \right)^d \le 2^{d+1} \delta^d \mu \left(\bigcap_{j \le i} A_j^c \right). \tag{11}$$

Lemma 1 follows from (9,10,11).

5. Proof of Theorem 1.

The arguments of this section are te same as the arguments from the variant of the Lovasz locall lemma used in [7].

We shall prove by induction that the inequality

$$\mu\left(\bigcap_{j\le i} A_j^c\right) \ge \frac{1}{2}\mu\left(\bigcap_{j\le i-h} A_j^c\right) > 0, \tag{12}$$

holds for all natural i.

1. The base of induction. For $i \le 0$ we define $A_i^c = [0,1]^d$. Then the statement is trivial for $i \le 0$. We shall check (12) for $0 \le i \le h$. It is sufficient to see that

$$\mu\left(\bigcap_{1\le j\le h} A_j^c\right) \ge \frac{1}{2}. \tag{13}$$

But

$$\mu\left(\bigcap_{1\le j\le h} A_j^c\right) \ge 1 - \sum_{j=1}^{h} \mu(A_j).$$

We must take into account (7). Now (13) follows from (3).

2. The inductive step. We suppose (12) to be true for all $i \le t$.

We have

$$\bigcap_{j\le t+1} A_j^c = \left(\ldots\left(\left(\bigcap_{j\le t+1-h} A_j^c\right) \setminus A_{t+1-h+1}\right) \setminus \cdots\right) \setminus A_{t+1}.$$

Then

$$\mu\left(\bigcap_{j\le t+1} A_j^c\right) \ge \mu\left(\bigcap_{j\le t+1-h} A_j^c\right) - \sum_{v=1}^{h} \mu\left(A_{t-h+1+v} \bigcap \left(\bigcap_{j\le t+1-h} A_j^c\right)\right). \tag{14}$$

Note that for the values of v under consideration we have $t+1-h \ge t+1+v-2h$. It means that

$$\bigcap_{j\le t+1-h} A_j^c \subseteq \bigcap_{j\le t+1+v-2h} A_j^c.$$

Hence by Lemma 1

$$\mu\left(A_{t-h+1+v} \bigcap \left(\bigcap_{j\le t+1-h} A_j^c\right)\right) \le \mu\left(A_{t-h+1+v} \bigcap \left(\bigcap_{j\le t+1+v-2h} A_j^c\right)\right) \le$$

$$\leq \delta_1 \mu \left(\bigcap_{j \leq t+1+v-2h} A_j^c \right).$$

So for $1 \leq v \leq h$ we have

$$\mu \left(A_{t-h+1+v} \bigcap \left(\bigcap_{j \leq t+1-h} A_j^c \right) \right) \leq \delta_1 \mu \left(\bigcap_{1 \leq t+1-2h+v} A_j^c \right) \leq \delta_1 \mu \left(\bigcap_{1 \leq t+2-2h} A_j^c \right).$$

But $h \geq 2$ and from our inductive hypothesis for $t+2-h$ we have

$$\mu \left(\bigcap_{j \leq t+1-h} A_j^c \right) \geq \mu \left(\bigcap_{j \leq t+2-h} A_j^c \right) \geq \frac{1}{2} \mu \left(\bigcap_{j \leq t+2-2h} A_j^c \right)$$

and hence

$$\mu \left(\bigcap_{j \leq t+2-2h} A_j^c \right) \leq 2\mu \left(\bigcap_{j \leq t+1-h} A_j^c \right).$$

Now

$$\mu \left(A_{t+1+h-v} \bigcap \left(\bigcap_{j \leq t+1-h} A_j^c \right) \right) \leq 2\delta_1 \mu \left(\bigcap_{1 \leq t+1-h} A_j^c \right). \tag{15}$$

So from (14,15) we have

$$\mu \left(\bigcap_{j \leq t+1} A_j^c \right) \geq (1 - 2\delta_1 h)\, \mu \left(\bigcap_{j \leq t+1-h} A_j^c \right)$$

and (12) follows from the inequality (4). Now from (12) and the compactness of the sets A_i^c Theorem 1 follows.

6. Comments on the proof of Theorem 2.

The proof of the first inequality from theorem 2 is quite similar to the proof of the Theorem 1. Instead of parameters δ, δ_1, h we should use

$$\delta' = \frac{1}{2^{13} d (M \log M)^{1/d}}, \;\; \delta_1' = 2^{4d+1} d^d (\delta')^d, \;\; h' = \lceil 2^{3d+3} M \log M \rceil.$$

The inequalities analogical to (3,4,5) are satisfied. We must deal with the sets $E'(j, \mathbf{a}) = S_j^{-1} E(j, \mathbf{a})$ and their covering by dyadic boxes A_j'. Then instead of (7) we get

$$\mu(A_j') \leq \left(\frac{2(\lceil \sqrt{d} \rceil + 1)\delta'}{t_j} \right)^d (\lceil \sqrt{d} t_j \rceil + 1)^d \leq (16\delta')^d.$$

instead of the Lemma 1 we get

Lemma 2. *Let* $\mu\left(\bigcap_{j\le i}(A'_j)^c\right)\ne 0$ *Then*

$$\mu\left(A'_{i+h}\bigcap(\bigcap_{j\le i}(A'_j)^c)\right)\le \delta'_1\mu\left(\bigcap_{j\le i}(A'_j)^c\right).$$

The sketch of the proof is as follows. For the number W'_ν of the boxes $E'(j+h,\mathbf{a})$ intersecting the box I_ν we have

$$W'_\nu\le (2^{-l_i}t_{i+h}+2\sqrt{d})^d$$

instead of (8). Then instead of (9) we get

$$\mu\left(A'_{i+h}\bigcap\left(\bigcap_{j\le i}(A'_j)^c\right)\right)\le (2\sqrt{d})^d\sum_{\nu=1}^{T_i}\frac{2^dW'_\nu}{2^{dl_{i+h}}}\le$$

$$\le 2^{2d}d^d\left(\sum_{\nu=1}^{T_i}\mu(I_\nu)\left(\frac{t_{i+h}}{2^{l_{i+h}}}\right)^d+\sum_{\nu=1}^{T_i}\left(\frac{1}{2^{l_{i+h}-1}}\right)^d\right).$$

From these inequalities we deduce Lemma 2. Now the proof of the Theorem 2 follows section 5 word by word.

References

[1] Erdös P. Repartition mod 1. // Lecture Notes in Math. 475, Springer-Verlag, N.Y., 1975.

[2] Pollington A.D. On the density of the sequence $\{n_k\theta\}$. // Illinois J. Math. 23 (1979), No. 4, 511-515.

[3] de Mathan B. Numbers contravening a condition in density modulo 1. // Acta Math. Acad. Sci. Hungar. 36 (1980), 237-241.

[4] Katznelson Y. Chromatic numbers of Cayley graphs on $\mathbb{Z}$ and recurrence. // Combinatorica 21 (2001), 211-219.

[5] Akhunzhanov R.K., Moshchevitin N.G. On the chromatic number of the distance graph associated with a lacunary sequence. // Doklady Akademii Nauk. Ross. 397 (2004), 295-296.

[6] Dubickas A. On the fractional parts of lacunary sequences. // Mathematica Scand. 99 (2006), 136-146.

[7] Peres Y., Schlag W. Two Erdös problems on lacunary sequences: chromatic numbers and Diophantine approximations. // Preprint, available at: arXiv:0706.0223v1 [math.CO] 1Jun2007

[8] Dubickas A. On the distribution of powers of a complex number., 2007, preprint.

author: Nikolay Moshchevitin

e-mail: moshchevitin@mech.math.msu.su, moshchevitin@rambler.ru